

AD-A185 625

THE PARADOXICAL ASYMPTOTIC STATUS OF MASSLESS SPRINGS

1/1

(U) MARYLAND UNIV COLLEGE PARK DEPT OF MATHEMATICS

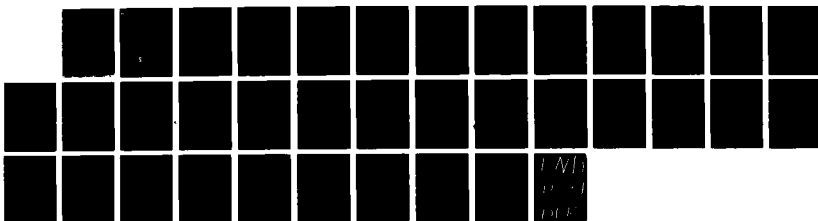
S S ANTMAN MAR 87 MD87-11-55A AFOSR-TR-87-1380

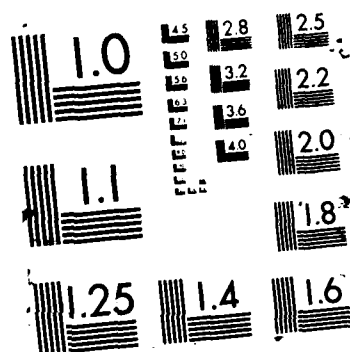
UNCLASSIFIED

#AFOSR-87-0073

F/G 20/11

NL





DTIC

TATION PAGE

Form Approved  
OMB No. 0704-0188

1a. REPORT

AD-A185 625

2a. SECURITY

2b. DECLASSIFICATION/DOWNGRADING SCHEDULE

4. PERFORMING ORGANIZATION REPORT NUMBER(S)

1b. RESTRICTIVE MARKINGS

3. DISTRIBUTION/AVAILABILITY OF REPORT

Approved for public release;  
distribution unlimited.

5. MONITORING ORGANIZATION REPORT NUMBER(S)

AFOSR-TR- 87-1380

6a. NAME OF PERFORMING ORGANIZATION

6b. OFFICE SYMBOL  
(if applicable)

7a. NAME OF MONITORING ORGANIZATION

6c. ADDRESS (City, State, and ZIP Code)

7b. ADDRESS (City, State, and ZIP Code)

8a. NAME OF FUNDING/SPONSORING  
ORGANIZATION8b. OFFICE SYMBOL  
(if applicable)

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER

AFOSR-87-0073

8c. ADDRESS (City, State, and ZIP Code)

10. SOURCE OF FUNDING NUMBERS

PROGRAM  
ELEMENT NO.PROJECT  
NO.TASK  
NO.WORK UNIT  
ACCESSION NO.

11. TITLE (Include Security Classification)

12. PERSONAL AUTHOR(S)

13a. TYPE OF REPORT

13b. TIME COVERED

FROM \_\_\_\_\_ TO \_\_\_\_\_

14. DATE OF REPORT (Year, Month, Day)

15. PAGE COUNT

16. SUPPLEMENTARY NOTATION

17. COSATI CODES

FIELD

GROUP

SUB-GROUP

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

DTIC  
ELECTE  
OCT 26 1987  
S  
D

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT

☐ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS

21. ABSTRACT SECURITY CLASSIFICATION

22a. NAME OF RESPONSIBLE INDIVIDUAL

22b. TELEPHONE (Include Area Code)

22c. OFFICE SYMBOL

The Paradoxical Asymptotic Status of Massless Springs

Stuart S. Antman

Department of Mathematics  
University of Maryland  
College Park, MD 20742

MD87-11-SSA

TR87-11

March 1987



Accession For	
NTIS CRA&I	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By _____	
Distribution /	
Availability Codes	
Dist	Avail and/or Special
A-1	

## 1. Introduction

→ The most fundamental problem in the entire theory of oscillations is to describe the motion of a mass point, the tip mass attached to a spring. Within the classical theory of particle mechanics, the spring is regarded as massless, so that it serves only to transmit a force to the tip mass. This force typically depends on the position and velocity of the tip mass in perhaps a nonlinear way. In this case, the motion is governed by an autonomous ordinary differential equation. On the other hand, if the spring has mass, then its motion as a continuum is coupled to that of the tip mass. If the spring has a nonlinear constitutive equation, then the analysis of the resulting motion, governed by partial differential equations, can be formidable indeed.

<sup>So, this</sup>  
In this paper we study the motion of both tip mass and spring when the mass density of the spring is small and when its constitutive equation describes nonlinearly elastic and viscoelastic materials. Although these constitutive equations do not account for past history, we nevertheless prove that in the formal limit as the spring's mass density goes to zero the equation for the tip mass is an ordinary differential equation for elastic springs, but is generally not so for viscoelastic springs. For the latter, it is typically an ordinary-functional differential equation in which the force law for the massless spring depends upon the past history of the motion of the tip mass. We furnish restrictions on the material properties and initial conditions for the motion to be governed by an ordinary differential equation in the usual form. Although these restrictions are quite special, it is fortunate

that they are often encountered in practice.

A rigorous asymptotic analysis of the full system of equations for nonlinearly viscoelastic springs is carried out in [1]. It shows that the equations studied here constitute those for the leading term of the regular expansion. The dissipative mechanism associated with the viscoelasticity plays a crucial role in this analysis. A comparable justification of the asymptotic status of elastic springs is not possible. The full asymptotic analysis relies on delicate estimates for quasilinear parabolic equations.

The mathematical tools used in the present paper are far less esoteric. The novelty of our results lies not in the analysis, but in the surprising structure of the equations and in the methods for manipulating them. Nevertheless, the finding of paradoxical behavior here would be vacuous, were it not backed up by rigorous analysis.

This paper has two goals: (i) to describe the surprisingly complicated behavior of massless viscoelastic springs; (ii) to serve as the first step in a development (pushed much further in [1]) of effective techniques for the treatment of the dynamics of structures undergoing large motions and deformations.

Notation. We denote partial derivatives by subscripts and ordinary derivatives by primes. To avoid ambiguities we carefully distinguish between a function and its values. Thus the function  $N$ , appearing in Section 3, is formally presented as

$$(1.1) \quad [0,1] \times (-1,\infty) \ni (x,y) \longmapsto N(x,y) \in \mathbb{R}.$$

Then  $N_x(x,y)$  is its partial derivative with respect to its first argument evaluated at  $(x,y)$ . We denote by  $x \mapsto N(x,y)$  or  $N(\cdot,y)$

the function of  $x$  obtained from  $N$  by fixing  $y$ . In our notation the term  $N(s, w_s(s, t))_s$  is unambiguously given by the chain rule as

$$(1.2) \quad N_x(s, w_s(s, t)) + N_y(s, w_s(s, t))w_{ss}(s, t).$$

Were we to follow the common practice of replacing the  $x$  in (1.1) with  $s$ , then the meaning of  $N_s(s, w_s(s, t))$  would no longer be clear.

## 2. Formulation of the Governing Equations

We define a spring to be a one-dimensional solid body whose configuration is confined to a line. Since the treatment of gravity or of other natural body forces offers neither challenge nor insight, we take these forces to be zero. We may accordingly regard the spring as confined to a straight smooth horizontal groove. One end of the spring is fixed at a point taken to be the origin of coordinates of a line along the groove. A particle of mass  $m$  is attached to the other end of the spring.

The natural length of the spring is taken to be 1. We identify a typical material point of the spring with its distance  $s$  from the fixed end in its natural state. In a motion of the spring-mass system, at time  $t$  the material point  $s$  occupies the position at distance  $s + w(s, t)$  from the fixed end.  $w(s, t)$  is the displacement of  $s$  at time  $t$ . We require that

$$(2.1) \quad w_s(s, t) > -1, \quad \forall s \in [0, 1], \quad \forall t \geq 0$$

to ensure that the local ratio of deformed to natural length never be reduced to zero. The requirement that the end  $s = 0$  be fixed is expressed by

$$(2.2) \quad w(0, t) = 0.$$

We impose the initial condition

$$(2.3a, b) \quad w(s, 0) = \omega(s), \quad w_t(s, 0) = \psi(s) \quad \text{for } s \in [0, 1].$$

Here  $\omega$  and  $\psi$  are prescribed, continuously differentiable functions on  $[0, 1]$  with  $\omega'(s) > -1$  for all  $s \in [0, 1]$  for consistency with (2.1) and with  $\psi(0) = 0$  for consistency with (2.2).

Let  $-n(s, t)$  be the (component of the) contact force (along



the groove) exerted on the material of  $[s,1]$  by that of  $[0,s]$  at time  $t$ . Let  $\varepsilon\rho(s)$  denote the mass density of the spring per unit reference length at  $s$ .  $\rho$  is assumed to be twice continuously differentiable and positive on  $[0,1]$ .  $\varepsilon$  is a small positive number. We assume that there are no externally applied forces acting on the spring-mass system (since their treatment would be routine). If  $w$  is sufficiently regular, the linear momentum balance yields the integral form of the equations of motion:

$$(2.4) \quad -n(s,t) = \int_s^1 \varepsilon\rho(\xi)w_{tt}(\xi,t)d\xi + mw_{tt}(1,t),$$

from which we immediately deduce the differential equation of motion:

$$(2.5) \quad \varepsilon\rho(s)w_{tt}(s,t) = n_s(s,t)$$

and the boundary condition at  $s = 1$ :

$$(2.6) \quad mw_{tt}(1,t) = -n(1,t).$$

The properties of the material at  $s$  are specified by giving the dependence of  $n(s,t)$  on the history of  $w_s(s,\cdot)$  up to time  $t$  by means of a suitable constitutive functional. Most of our interest will be directed to viscoelastic springs of differential type 1 whose constitutive equations are expressed in terms of given constitutive functions

$$(2.7) \quad [0,1] \times (-1,\infty) \times (-\infty,\infty) \ni (x,y,z) \mapsto N(x,y,z) \in \mathbb{R}$$

by

$$(2.8) \quad n(s,t) = N(s, w_s(s,t), w_{st}(s,t)).$$

(We systematically use the arguments introduced in (2.7) to iden-

tify the various partial derivatives of  $N$ .) For contrast we shall discuss elastic springs defined by (2.7), (2.8) with

$$(2.9) \quad N_z = 0.$$

To ensure that an increase in extension (strain)  $w_s$  for fixed rate of strain  $w_{st}$  be accompanied by a corresponding increase in the tensile force, we require that

$$(2.10) \quad N_y > 0.$$

To ensure that a total compression be accompanied by an infinite compressive force and that an infinite extension be accompanied by an infinite tensile force we require that

$$(2.11) \quad N(x, y, z) \rightarrow \begin{cases} \infty \\ -\infty \end{cases} \text{ as } y \rightarrow \begin{cases} \infty \\ -1 \end{cases} \text{ for } x \in [0, 1] \\ \text{and for } z \text{ bounded.}$$

To ensure that all motions of viscoelastic springs are dissipative we require that

$$(2.12) \quad N_z \geq \text{const} > 0.$$

For simplicity we assume that  $N$  is continuously differentiable and that  $N(x, \cdot, \cdot)$  is infinitely differentiable.

Assumption (2.12) implies that  $z \mapsto N(x, y, z)$  has an inverse

$$(2.13) \quad n \mapsto Z(x, y, n).$$

It follows from (2.7) and (2.12) that

$$(2.14) \quad Z(x, y, n) \longrightarrow \pm \infty \text{ as } n \longrightarrow \pm \infty,$$

from (2.12) that

$$(2.15) \quad Z_n > 0,$$

and from (2.10) and (2.15) that

$$(2.16) \quad Z_y < 0.$$

Now let  $n = N(x, y, z)$  or, equivalently,  $z = Z(x, y, n)$ . Suppose that  $y \rightarrow -1$  or  $\infty$ , while  $n$  remains bounded. Then (2.11) and (2.12) imply that  $z \rightarrow \infty$  or  $-\infty$ , respectively. Thus (2.11) and (2.12) also imply that

$$(2.17) \quad Z(x, y, n) \rightarrow \begin{cases} \infty \\ -\infty \end{cases} \text{ as } y \rightarrow \begin{cases} -1 \\ \infty \end{cases}$$

for  $n$  bounded.

The initial-boundary value problem for  $w$  is (2.2), (2.3), (2.5), (2.6), (2.8). The reduced problem corresponding to it, obtained by setting  $\varepsilon = 0$  in these equations, is

$$(2.18) \quad N(s, w_s(s, t), w_{st}(s, t)) = N(1, w_s(1, t), w_{st}(1, t)),$$

$$(2.19) \quad w(0, t) = 0,$$

$$(2.20) \quad mw_{tt}(1, t) + N(1, w_s(1, t), w_{st}(1, t)) = 0,$$

$$(2.21a, b) \quad w(s, 0) = \omega(s), \quad w_t(1, 0) = \psi(1).$$

In (2.21) we retain only those initial data needed to ensure that (2.18) - (2.21) has a unique solution for small time. For compatibility of (2.19) and (2.21a, b) we require that  $\omega(0) = 0$ . Moreover, (2.18) and (2.21) yield

$$(2.22) \quad N(s, \omega'(s), w_{st}(s, 0)) = N(1, \omega'(1), w_{st}(1, 0)),$$

which is equivalent to

$$(2.23) \quad w_{st}(s, 0) = Z(s, \omega'(s), N(1, \omega'(1), w_{st}(1, 0))).$$

Since (2.19) implies that

$$(2.24) \quad w_t(0, 0) = 0,$$

we obtain from (2.23) that

$$(2.25) \quad w_t(s,0) = \int_0^s Z(\xi, \omega'(\xi), N(1, \omega'(1), w_{st}(1,0))) d\xi.$$

There is no reason to expect this  $w_t(\cdot, 0)$  to agree with  $\psi$  prescribed in (2.3b). In fact, setting  $s = 1$  in (2.25) and using (2.21b) we obtain

$$(2.26) \quad \psi(1) = \int_0^1 Z(\xi, \omega'(\xi), N(1, \omega'(1), w_{st}(1,0))) d\xi,$$

which we regard as an equation for  $w_{st}(1,0)$  in terms of the initial data of (2.21). (Equation (2.26) can be uniquely solved for  $w_{st}(1,0)$ , because the right side of (2.26) is a strictly increasing function of  $w_{st}(1,0)$  that ranges over  $(-\infty, \infty)$  with  $w_{st}(1,0)$ .) This  $w_{st}(1,0)$  need not agree with  $\psi'(1)$  of (2.3b). Once  $w_{st}(1,0)$  is found, we can substitute it into (2.23) to produce  $w_{st}(s,0)$ , which need not agree with  $\psi(s)$  of (2.3b).

Remark. Every three-dimensional interpretation of our variables shows that it is reasonable to assume that  $N$  and  $\epsilon\rho$  are each proportional to the cross-sectional area of the spring. Thus we might be led to replace  $N$  with  $\epsilon N^*$ , with  $\epsilon$  interpreted as the cross-sectional area. Such a parametrization would significantly alter the character of our equations. To avoid confusion on this question, it is helpful to regard the properties of the spring, namely its density and constitutive function, as fixed and to parametrize the problem by the magnitude of the tip mass. (In our analysis the small parameter  $\epsilon$  should be interpreted as proportional to the ratio of the total mass  $\epsilon \int_0^1 \rho ds$  of the spring to the mass  $m$ .)

A simple physical argument reinforces these remarks. Consider the equilibrium of a spring subject only to a constant end load  $n(1,t) = \lambda$ . Then the equilibrium version of (2.3) would imply that  $n(s,t) = \lambda$  for all  $s$ . Thus no introduction of a small parameter  $\epsilon$  through a constitutive equation in the form

$$(2.27) \quad n(s,t) = \epsilon N^*(s, w_s(s,t), w_{st}(s,t))$$

could change the value of the tension  $n(s,t)$  (but of course any change in  $\lambda$  would do so). If (2.27) is used, then the dependence of  $w_s$  on  $\epsilon$ , captured by an appropriate scaling, would have to accommodate the weakening of the material. The scaling used would have to depend on  $N^*$ . The adoption of (2.27) in our problem would force us to confront the dynamical analog of this difficulty. The analysis of [1] gives a precise standing to the reduced problem (2.18) - (2.21); no attempt has been made to justify problems based on (2.27).

### 3. The Reduced Problem for Elastic Springs

We begin our study of the reduced problem in the degenerate case of elastic springs, which have constitutive equations of the form (2.8) with  $N$  independent of its last argument:  $N_z = 0$ . The solutions of the reduced equations for elastic springs have peculiarities not shared by those for other materials.

In view of (2.10), (2.11),  $N(x, \cdot)$  has an inverse  $(-\infty, \infty) \ni n \mapsto Y(x, n) \in (-1, \infty)$ . Let us assume that  $N(x, 0) = 0$  so that  $Y(x, 0) = 0$ . Then the reduced equation (2.14) subject to (2.9) is equivalent to

$$(3.1) \quad w_s(s, t) = Y(s, N(1, w_s(1, t))).$$

Our goal is to find an equation for  $w(1, \cdot)$ , which governs the motion of the end mass, when  $w$  satisfies (3.1), (2.19) - (2.21). Equation (2.20), which we rewrite as

$$(3.2) \quad mw_{tt}(1, t) + N(1, w_s(1, t)) = 0,$$

is the natural place to seek such an equation, but is unsuitable as it stands because  $w_s(1, t)$  is not related to  $w(1, \cdot)$ . We now obtain such a relation.

From (3.1) and (2.19) we obtain

$$(3.3) \quad w(s, t) = \int_0^s Y(\xi, N(1, w_s(1, t))) d\xi,$$

$$(3.4) \quad w(1, t) = \int_0^1 Y(\xi, N(1, w_s(1, t))) d\xi = J(w_s(1, t)).$$

The properties of  $N$  show that  $J$  increases strictly from  $-1$  to  $\infty$  as its argument increases from  $-1$  to  $\infty$ . Thus (3.4) is equivalent to

$$(3.5) \quad w_s(1,t) = J^{-1}(w(1,t)).$$

The substitution of (3.5) into (3.2) yields an ordinary differential equation for  $w(1,\cdot)$ :

$$(3.6) \quad mw_{tt}(1,t) + N(1, J^{-1}(w(1,t))) = 0.$$

Initial conditions for  $w(1,\cdot)$  are given by (2.21a).

Note that (3.4) implies that  $w(1,t) > -1$ . If the spring is uniform, i.e., if  $N_x = 0$ , then  $J$  is just the identity. The properties of  $w(1,\cdot)$  are readily found by a phase-plane analysis of (3.6). The phase portrait consists of closed orbits around the origin. By substituting (3.5) into (3.3) we obtain  $w$  in terms of  $w(1,\cdot)$ :

$$(3.7) \quad w(s,t) = \int_0^s Y(\xi, N(1, J^{-1}(w(1,t)))) d\xi,$$

which reduces to

$$(3.8) \quad w(s,t) = sw(1,t)$$

if the spring is uniform.

Note that in general  $w$  does not satisfy even (2.21a). Equation (3.7) shows that (2.21a) holds if and only if

$$(3.9) \quad \omega(s) = \int_0^s Y(\xi, N(1, J^{-1}(w(1)))) d\xi.$$

From (3.7) we obtain

$$(3.10) \quad w_t(s,0) = \psi(1) \int_0^s \frac{\partial}{\partial y} Y(\xi, N(1, J^{-1}(y))) \big|_{y=\omega(1)} dy.$$

In the initial state given by (3.9), (3.10) with  $\psi(1) = 0$ , the spring is at rest under the prescribed end displacement  $\omega(1)$ . Of all initial configuration at rest, these, parameterized by  $\omega(1)$ ,

are the only ones that can be maintained with zero body force.

Note that for uniform springs, we find from (3.8) that (3.9),

(3.10) reduce to

$$(3.11) \quad \omega(s) = s\omega(1), \quad w_t(s,0) = s\psi(1),$$

which have particularly simple interpretations.



#### 4. The Reduced Problem for Viscoelastic Springs

We now assume that  $N$  satisfies (2.12).

If the spring is uniform, i.e., if  $N_x = 0$ , then we immediately read off a solution to (2.18):

$$(4.1) \quad w_s(s, t) = w_s(1, t).$$

It follows from (2.19) that

$$(4.2) \quad w(s, t) = w_s(1, t)s, \quad w(1, t) = w_s(1, t), \quad w_t(1, t) = w_{st}(1, t).$$

Substituting (4.2) into (2.20) we obtain the following ordinary differential equation for the tip mass

$$(4.3) \quad mw_{tt}(1, t) + N(1, w(1, t), w_t(1, t)) = 0.$$

In general, the solution (4.1) satisfies the initial condition (2.21a) if and only if

$$(4.4) \quad \omega(s) = \omega'(1)s.$$

The initial condition (2.21b) is discussed in the paragraph containing (2.22). The interpretation of (4.4) is the same as that for (3.9).

We now turn to the construction of a solution  $w$  of the reduced problem when the spring is not uniform or when arbitrary initial conditions of the form (2.21) are imposed.

Let us set

$$(4.5) \quad \nu(t) = N(1, w_s(1, t), w_{st}(1, t)).$$

Then (2.18) implies that

$$(4.6) \quad w_{st}(s, t) = Z(s, w_s(s, t), \nu(t)).$$

We rewrite (2.20) as

$$(4.7) \quad mw_{tt}(1,t) + \nu(t) = 0.$$

From (2.19) we get

$$(4.8) \quad w(1,t) = \int_0^1 w_s(s,t) ds$$

so that (4.6) yields

$$(4.9) \quad w_t(1,t) = \int_0^1 Z(s, w_s(s,t), \nu(t)) ds.$$

We now differentiate (4.9) with respect to  $t$ , use (4.7), rearrange the resulting equation, and use (4.6) to obtain

$$(4.10) \quad \nu_t(t) = - \frac{\nu(t)/m + \int_0^1 Z_y Z dx}{\int_0^1 Z_n ds}$$

where the arguments of  $Z, Z_y, Z_n$  are  $s, w_s(s,t), \nu(t)$ . Equations (4.6) and (4.10) are a pair of ordinary differential equations for  $t \mapsto w_s(\cdot, t), \nu(t)$ . They are subject to the initial conditions

$$(4.11a,b) \quad w_s(s,0) = \omega(s), \quad \nu(0) = N(1, \omega'(1), w_{st}(1,0))$$

where  $w_{st}(1,0)$  satisfies (2.26). In a completely standard way, the Contraction Mapping Principle implies

**4.12. Theorem.** There is a number  $t^+$  such (4.6), (4.10), (4.11) has a unique, continuously differentiable solution  $(0, t^+) \ni t \mapsto w_s(\cdot, t), \nu(t) \in C^0([0,1], (-1, \infty)) \times \mathbb{R}$ .

**Remark.** Condition (2.12) ensures that (4.9) can be uniquely solved for  $\nu(t)$ :

$$(4.13) \quad \nu(t) = \nu^*(w_s(\cdot, t), w_t(1, t)).$$

If we replace  $\nu$  in (4.6) and (4.7) with (4.13) we obtain a

system of two first-order ordinary differential equations for  $w_s(\cdot, t)$  and  $w_t(1, t)$  which can be used in place of (4.8) and (4.10) to produce a theorem analogous to 4.12. Note that (4.7) and Theorem 4.12 imply that  $w_{tt}(1, \cdot)$  is continuously differentiable on  $[0, t^+)$ .

Now let us regard (4.6) as an ordinary differential equation for  $w_s(s, \cdot)$  when  $\nu$  is regarded as given. Since  $Z(x, \cdot, n)$  is continuously differentiable, the continuation theory for ordinary differential equations satisfying the Carathéodory conditions (cf. [3, Chap. 2], [4, Sec. I.5]) imply that on any interval  $[0, t^{++})$ , possibly containing  $[0, t^+]$ , on which  $\nu$  is bounded (and measurable), equation (4.6) has a unique absolutely continuous solution  $w_s(s, \cdot)$  satisfying (4.11a) as long as  $w_s(s, \cdot)$  is confined to  $(-1, \infty)$ . Now the parameter  $s$  appears in the initial value problem (4.6), (4.11a) in the initial datum  $\omega(s)$  and in the dependence of  $Z$  on its first argument. By the continuous dependence of solutions on the data (cf. [3, Theorem 2.4.2]) we find that  $w_s$  depends continuously on  $s$  for  $t \in [0, t^{++})$  as long as  $w_s(s, \cdot)$  stays in  $(-1, \infty)$ . But condition (2.17) ensures that  $w_s(s, \cdot)$  cannot leave  $(-1, \infty)$  on  $[0, t^{++})$  as a sketch of the slope field  $(t, y) \mapsto Z(s, y, \nu(t))$  immediately shows. Hence we conclude that

4.14. Proposition. On any interval  $[0, t^{++})$  on which  $\nu$  is bounded (and measurable), problem (4.6), (4.11a) has a unique solution  $w_s(s, \cdot)$  with  $w_s$  continuous in  $s$ .

It then follows from (4.10) that  $\nu$  must be continuous on such an interval. Let us combine (4.6) and (4.7) to obtain

$$(4.15) \quad w_{st}(s, t) = Z(s, w_s(s, t), -mw_{tt}(1, t)).$$

We can now invoke the continuous dependence of solutions on the equation (cf. [5, Section 3.3]) to obtain

4.16 Theorem. On any interval  $[0, t^{++})$  on which  $\nu = -mw_{tt}(1, \cdot)$  is bounded and measurable, it is continuous. On such an interval, problem (4.13), (4.11a) has a unique continuously differentiable solution  $w_s(s, \cdot)$  with the properties that  $w_s(s, t) \in (-1, \infty)$  for  $0 \leq t < t^{++}$ ,  $w_s$  depends continuously on  $s$ , and  $w_s(s, t)$  depends continuously on the restriction of  $w_{tt}(1, \cdot)$  and thus of  $w(1, \cdot)$  to  $[0, t]$  for  $0 \leq t < t^{++}$ . A fortiori,  $w_s(1, t)$  and  $w_{st}(1, t)$  depend continuously on the past history of  $w(1, \cdot)$ .

We now find conditions ensuring that  $t^{++} = \infty$ . Let

$$(4.17) \quad E(x, y) = \int_0^y N(x, \eta, 0) d\eta.$$

Condition (2.10) ensures that  $E(x, \cdot)$  is strictly convex and (2.11) that  $E(x, y) \rightarrow \infty$  as  $y \rightarrow \infty$ . We strengthen (2.11) by requiring that

$$(4.18) \quad E(x, y) \rightarrow \infty \text{ as } y \rightarrow -1, \infty.$$

Let us multiply (2.18) by  $w_s(s, t)$  and use (4.5), (4.7), (4.18) to write the resulting equation as

$$(4.19) \quad \begin{aligned} & E_y(s, w_s(s, t)) w_{st}(s, t) \\ & + [N(s, w_s(s, t), w_{st}(s, t)) - N(s, w_s(s, t), 0)] w_{st}(s, t) \\ & = \nu(t) w_{st}(s, t) = -mw_{tt}(1, t) w_{st}(s, t). \end{aligned}$$

Integrating (4.19) with respect to  $s$  over  $[0, 1]$  and using (4.8) we obtain the energy equation

$$\begin{aligned}
 (4.20) \quad & \frac{d}{dt} \left\{ \frac{m}{2} w_t(1,t)^2 + \int_0^1 E(s, w_s(s,t)) ds \right\} \\
 &= - \int_0^1 \left[ N(s, w_s(s,t), w_{st}(s,t)) - N(s, w_s(s,t), 0) \right] w_{st}(s,t) ds.
 \end{aligned}$$

The first term in the braces on the left side of (4.20) is the kinetic energy of the tip mass and the second term is the potential energy of the spring. In the reduced problem, the spring has no kinetic energy. Since the right side of (4.20) cannot be positive by (2.12) we obtain

$$(4.21) \quad \frac{m}{2} w_t(1,t)^2 + \int_0^1 E(s, w_s(s,t)) ds \leq C.$$

Here and below,  $C$  represents a positive constant independent of  $t$ , depending only on the data of the problem. Thus for  $t \in [0, t^{++})$ ,

$$(4.22) \quad |w_t(1,t)| \leq C, \quad |w(1,t)| \leq C(1+t).$$

Next

$$(4.23) \quad w(1,t) > -1 \quad \text{for } t \text{ in any compact subinterval of } [0, t^{++}),$$

for if not, there would be a  $\tau \in (0, t^{++})$  such that

$$(4.24) \quad w(1,t) = \int_0^1 w_s(s,t) ds \searrow -1 \quad \text{as } t \nearrow \tau.$$

Thus  $w_s(\cdot, t)$  would converge to  $-1$  in measure as  $t \nearrow \tau$ . But this result contradicts (4.18) and (4.21).

Now we show that  $\| \cdot \|$  cannot blow up in finite time. If  $\nu(t) \rightarrow \infty$  as  $t \rightarrow t^{++}$ , then (2.18) and (4.5) would imply that for

each  $s$  in  $[0,1]$

$$(4.25) \quad w_s(s,t) \longrightarrow \lambda \quad \text{or} \quad w_{s+}(s,t) \longrightarrow \lambda \quad \text{as} \quad t \longrightarrow t^{++}$$

as a consequence of (2.11), (2.12). But (4.8) would then imply that

$$(4.26) \quad w(1,t) \longrightarrow \infty \quad \text{or} \quad w_t(1,t) \longrightarrow \infty \quad \text{as} \quad t \longrightarrow \infty$$

in contradiction to (4.22). If  $v(t) \rightarrow -\infty$  as  $t \rightarrow t^{++}$ , then analogously for each  $s$  in  $[0,1]$

$$(4.27a,b) \quad w_s(s,t) \longrightarrow -1 \quad \text{or} \quad w_{st}(s,t) \longrightarrow -\infty \quad \text{as} \quad t \longrightarrow t^{++}.$$

Now (4.27b) cannot hold on a set of positive measure since it would imply by (4.8) that  $w_t(1,t) \rightarrow -\infty$  in violation of (4.22).

Thus (4.27a) would have to hold for almost all  $s$  in  $[0,1]$  implying by (4.8) that  $w(1,t) \rightarrow -1$  as  $t \rightarrow t^{++}$ . This result is incompatible with (4.23) if  $t^{++} < \infty$ . Hence we have

4.29. Theorem. Let (4.18) hold. Then Theorems 4.12 and 4.16 hold with  $t^+ = t^{++} = \infty$ .

Theorem 4.16 says that  $w_s(1,t)$  depends on the past history of  $w_s(1,\cdot)$ . If this dependence is substituted into (2.20), we might well get a functional-differential equation for the tip mass. Could the abstraction (in the form of the contraction mapping principle) underlying the proof of Theorem 4.16 obscure the possibility that  $w_s(1,t)$  might depend only on the present value  $w(1,t)$ ? What are the conditions for this to happen? I.e., what are the conditions for the motion of the end mass to be governed by an ordinary differential equation? We address these questions in the rest of this paper. We begin by studying an illuminating example.

## 5. Example Linearly Viscoelastic Springs

We illustrate the results of Section 4 with the reduced problem for linearly viscoelastic springs for which  $N$  has the form

$$(5.1) \quad N(x, y, z) = p(x)z + q(x)y$$

where  $p$  and  $q$  are positive and continuous on  $[0, 1]$ . For this linear law, presumably valid for small motions, it is inappropriate and impossible to require that (2.11) hold.

Let

$$(5.2) \quad r(x) = q(x)/p(x).$$

Then we can write (2.18) as

$$(5.3) \quad \left[ e^{r(s)t} w_s(s, t) \right]_t = p(1)p(s)^{-1} e^{[r(s)-r(1)]t} \left[ e^{r(1)t} w_s(1, t) \right]_t$$

from which we ultimately obtain

$$(5.4a) \quad w(1, t) = f(t) + a w_s(1, t) - \int_0^1 b(s) \int_0^t e^{r(s)(\eta-t)} w_s(1, \eta) d\eta ds$$

where

$$(5.4b) \quad f(t) = \int_0^1 e^{-r(s)t} [\omega'(s) - p(1)p(s)^{-1} \omega'(1)] ds,$$

$$(5.4c) \quad a = \int_0^1 p(1)p(s)^{-1} ds,$$

$$(5.4d) \quad b(s) = p(1)p(s)^{-1} [r(s) - r(1)].$$

To convert (2.20) to an equation for  $w(1, \cdot)$  alone, we must solve (5.4) for  $w_s(1, t)$  in terms of  $w(1, \cdot)$ . If  $r$  is constant, then the solution is immediate. Otherwise, observing that the integral over  $(0, t)$  on the right side of (5.4a) is a convolution, we can accordingly solve it by the Laplace transform method

to obtain

$$(5.5) \quad w_s(w, t) = \int_0^t [w(q, \eta) - f(\eta)] g(t - \eta) d\eta$$

where  $g$  is the inverse Laplace transform of

$$(5.6) \quad \sigma \mapsto \left[ a - \int_0^t \frac{b(s) ds}{\sigma - r(s)} \right]^{-1}.$$

In order for (2.20) to be an ordinary differential equation for  $w(1, \cdot)$ , we must be able to express  $w_s(1, t)$  in terms of  $w(1, t)$ . Since the solution (5.5) is unique, this possibility can occur if and only if  $g$  is the Dirac delta, i.e., if and only if (5.6) is a constant function. This happens if and only if  $b = 0$ , i.e., if and only if  $r$  is a constant. (The constancy of  $r$  may be regarded as a reflection of the proportionality of the moduli  $p$  and  $q$  to a variable cross-sectional area of the spring.) Thus if  $r$  is not constant, then the reduced equation for the tip mass is that for a massless viscoelastic spring of memory type.

Even if  $r$  is constant, the resulting equation for  $w_0(1, \cdot)$  is not autonomous unless the transient  $f = 0$ , i.e., unless  $\omega(1) = a\omega'(1)$ . We can also determine conditions ensuring that the functional differential equation (occurring when  $r$  is not constant) is autonomous. A sufficient condition for  $f = 0$  is that

$$(5.7) \quad h(s) = \omega'(s) - p(1)p(s)^{-1}\omega'(1) = 0.$$

If  $r$  is monotone, (5.7) is also necessary. For if  $f = 0$ , then all its derivatives vanish at 0:

$$(5.8) \quad 0 = \int_0^1 r(s)^k h(s) ds = \int_{r(0)}^{r(1)} x^k h(\sigma(x)) \sigma'(x) dx, \quad k = 0, 1, \dots,$$



where  $\sigma$  is the inverse of  $r$ . Since the monomials  $x^k$  are complete (in  $L_2$ , say), equation (5.7) follows.

The development encompassing (5.1) - (5.4) can be carried over for nonlinear constitutive functions of the form

$$(5.9) \quad N(x, y, z) = F(x, y) {}_y z + r(x) F(x, y),$$

but such  $N$ 's cannot satisfy (2.11). We find that if  $r$  is constant, then the reduced equation for the end mass is an ordinary differential equation. (We of course cannot apply the Laplace transform to our nonlinear equation to effect a general solution of the analog of (5.4).)

## 6. Necessary Conditions for the Reduced Problem for the End Mass to be Governed by an Ordinary Differential Equation

The results of the last two sections suggest two questions about the behavior of the reduced problem for nonlinearly viscoelastic springs:

i) What conditions on  $N$  ensure that the reduced problem for the tip mass (generated by (2.20)) be governed by an ordinary differential equation (possibly with a nonautonomous transient term) for all initial data  $\omega, \psi$  (satisfying (2.26) at  $t = 0$ )?

ii) What conditions on  $N$  and the initial data ensure that this problem is governed by an autonomous ordinary differential equation for all values of  $\omega(1), \psi(1)$ ? (These data are those appropriate for the motion of a mass point on a massless spring.)

We obtain necessary conditions, which can be used to show that such ordinary differential equations cannot occur for most materials. We begin with question (i). Since we have more data to vary for this problem, we get much sharper restrictions.

Evaluating (2.18) at  $t = 0$  and using (2.3) we obtain

$$(6.1) \quad N(s, \omega'(s), \psi'(s)) = N(1, \omega'(1), \psi'(1)),$$

which is a compatibility condition for initial data (cf. (2.22)).

If (2.20) is to generate an ordinary differential equation for  $w(1, \cdot)$ , then  $w_s(1, t)$  and  $w_{st}(1, t)$  must be functions of  $w(1, t)$  and  $w_t(1, t)$ . Indeed, if the order of the equation is not to exceed 2, then this dependence must have the form

$$(6.2a) \quad w_s(1, t) = j(w(1, t), t),$$

$$(6.2b) \quad w_{st}(1, t) = j_w(w(1, t), t)w_t(1, t) + j_t(w(1, t), t)$$

where

$$(6.2c) \quad (-1, x) \cdot [0, x) \ni (w, \zeta) \longmapsto j(w, \zeta)$$

is a function to be determined. Let us set

$$(6.3) \quad \alpha \equiv \omega(1), \beta \equiv \psi(1).$$

By letting  $t \rightarrow 0$  in (6.2a,b) we obtain

$$(6.4) \quad \omega'(1) = j(\alpha, 0), \psi'(1) = j_w(\alpha, 0)\beta + j_\zeta(\alpha, 0).$$

Now we substitute (6.4) into (6.1), solve it for  $\psi'(s)$ , integrate the resulting equation with respect to  $s$  from 0 to 1, and use (2.19) to obtain

$$(6.5) \quad \beta = \int_0^1 Z(s, \omega'(s), N(1, j(\alpha, 0), j_w(\alpha, 0)\beta + j_\zeta(\alpha, 0))) ds.$$

(Using (6.5) we do not have to worry about how  $\psi'$  is determined from (6.1).) Equation (6.5) must hold identically for all sufficiently smooth functions  $\omega$  satisfying  $\omega(0) = 0$  and  $\omega' > -1$  and for all real numbers  $\beta$ . Thus the Gateaux differential (first variation) of the right side of (6.5) with respect to  $\omega$  in the direction  $\eta$  must vanish for all  $\eta$  with  $\eta(0) = 0$ :

$$(6.6) \quad 0 = \int_0^1 (Z_y \eta'(s) + Z_n [N_y^1 j_w(\alpha, 0) + N_z^1 (j_{ww}(\alpha, 0)\beta + j_{w\zeta}(\alpha, 0))] \eta(1)) ds.$$

Here and below we understand the arguments of  $Z$  to be those shown in (6.5) and we use  $N^1, N_y^1, \dots$  to stand for the values of  $N, N_y, \dots$  at the arguments of  $N$  shown in (6.5). By the fundamental lemma of the Calculus of Variations, we obtain from (6.6) that

$$(6.7) \quad Z_Y(s, \omega'(s), N^1) = Z_Y(1, \omega'(1), N^1), \quad \forall \omega', \alpha, \beta, s.$$

(The fundamental lemma also produces a natural boundary condition from (6.6).) The differentiation of (6.5) with respect to  $s$  yields

$$(6.8) \quad 1 = \left[ \int_0^1 Z_n ds \right] N_Z^1 j_w(\alpha, 0), \quad \forall \omega', \alpha, \beta.$$

Conditions (2.10) and (2.12) then imply that  $j_w(\alpha, 0)$  is everywhere positive. It follows that  $j_w(\alpha, 0)\beta$  and therefore  $N^1$  range over  $(-\infty, \infty)$  as  $\beta$  ranges over  $(-\infty, \infty)$ . Thus (6.7) implies that

$$(6.9) \quad Z_Y(s, \omega'(s), n) = Z_Y(1, \omega'(1), n), \quad \forall s, \omega', n.$$

Since  $\omega'(s)$  is arbitrary in  $(-1, \infty)$  and since  $s$  is arbitrary in  $[0, 1]$ , we conclude from (6.9) that

$$(6.10) \quad Z_Y(x, y, n) = Z_Y(1, \omega'(1), n), \quad \forall (x, y, n) \in [0, 1] \times (-1, \infty) \times (-\infty, \infty).$$

(Applying this condition to (5.1) we immediately obtain that  $q/p = \text{constant}$ , in complete agreement with the findings of Section 5.) But condition (2.17) implies that (6.10) cannot hold. Hence, our answer to question (i) is that there are no materials satisfying (2.10) - (2.12) for which the reduced problem for the tip mass is governed by an ordinary differential equation for all initial data.

We now turn to question (ii). We assume that  $N(x, 0, 0) = 0$ . We begin by studying (2.18) - (2.21) subject to small initial data of the form

$$(6.11) \quad w(s, 0) = \eta \omega_1(s), \quad w_t(1, 0) = \eta \psi_1(1)$$

where  $\eta$  is a real number small in absolute value. As in Section 4 we can use standard results from the theory of ordinary differ-

ential equations to show that (2.18) - (2.20), (6.11) have a solution of the form

$$(6.12) \quad w(s, t; \eta) = \eta w^1(s, t) + (\eta^2/2!) w^2(s, t) + \dots$$

We get the equations satisfied by  $w^r$  by substituting (6.12) into (2.18) - (2.20), (6.11), differentiating these equations  $r$  times with respect to  $\eta$ , and then setting  $\eta = 0$ .

Let us set  $\bar{N}_y(s) = N_y(s, 0, 0)$ , etc. Then  $w^1$  satisfies the linearization of (2.18), (6.11):

$$(6.13) \quad \bar{N}_z(s) w_{st}^1(s, t) + \bar{N}_y(s) w_s^1(s, t) = \bar{N}_z(1) w_{st}^1(1, t) + \bar{N}_y(1) w_s^1(1, t),$$

$$(6.14) \quad w^1(s, 0) = \omega_1(s).$$

The analysis of Section 5 shows that  $w_s^1(1, t)$  has the form  $j(w^1(1, t))$  if and only if

$$(6.15a, b) \quad \bar{N}_y(s)/\bar{N}_z(s) = r(\text{const.}), \quad \omega_1(1) = \omega_1'(1) \int_0^1 [\bar{N}_z(1)/\bar{N}_z(s)] ds.$$

In this case we readily find that

$$(6.16a, b) \quad w_s^1(s, t) = \gamma(s) w^1(1, t), \quad w^1(s, t) = w^1(1, t) \int_0^s \gamma(\xi) d\xi,$$

$$(6.16c) \quad \gamma(s) = \frac{\bar{N}_z(1)/\bar{N}_z(s)}{\int_0^1 [\bar{N}_z(1)/\bar{N}_z(\xi)] d\xi}.$$

Now  $w^2$  satisfies

$$(6.17a) \quad \left\{ \bar{N}_z(\xi) \left[ e^{rt} w_s^2(\xi, t) \right]_t + e^{rt} \left[ \bar{N}_{yy}(\xi) w_s^1(\xi, t)^2 + 2\bar{N}_{yz}(\xi) w_s^1(\xi, t) w_{st}^1(\xi, t) + \bar{N}_z(\xi) w_{st}^1(\xi, t)^2 \right] \right\} \Big|_{\xi=0}^{\xi=s} = 0,$$

$$(6.17b) \quad w^2(s,0) = 0.$$

Treating (6.17) in a manner just like that of Section 5, we find that  $w_s^2(1,t)$  depends only on  $w^2(1,t)$  and  $w^1(1,t)$  (without history terms or nonautonomous terms) if and only if

$$(6.18) \quad \int_0^1 \left\{ \left[ \bar{N}_{yy}(s)\gamma(s)^2 - \bar{N}_{yy}(1)\gamma(1)^2 \right] w^1(1,t)^2 \right. \\ \left. + 2 \left[ \bar{N}_{yz}(s)\gamma(s)^2 - \bar{N}_{yz}(1)\gamma(1)^2 \right] w^1(1,t) w_t^1(1,t) \right. \\ \left. + \left[ \bar{N}_{zz}(s)\gamma(s)^2 - \bar{N}_{zz}(1)\gamma(1)^2 \right] w_t^1(1,t)^2 \right\} ds = 0.$$

We let  $t \rightarrow 0$  in (6.18). Since  $\omega_1(1)$  and  $\psi_1(1)$  are independent, we obtain from (6.19) that

$$(6.19) \quad \bar{N}_{yy}(s)\gamma(s)^2, \bar{N}_{yz}(s)\gamma(s)^2, \bar{N}_{zz}(s)\gamma(s)^2 \text{ are constants.}$$

Equations (6.15) and (6.19) are two sets of necessary conditions for the reduced problem for the tip mass to be governed by an ordinary differential equation. It is clear how to continue this procedure ad infinitum.

An alternative global approach to problem (ii) based upon the formalism developed for problem (i) illuminates some of the issues involved, and it can directly produce the relationship between  $w_s(1,t)$  and  $w(1,t)$ . In this method we replace (6.2a) with

$$(6.20) \quad w_s(1,t) = j(w(1,t)), \quad j(0) = 0$$

because we require the resulting form of (2.20) to be autonomous. The second equation of (6.20) is consistent with the requirement that  $N(x,0,0) = 0$ . We adopt (6.3) and allow  $\sigma$  to depend upon  $\sigma$ :

$$(6.21) \quad \omega' = \alpha \chi(s, \alpha).$$

Then (6.5) reduces to

$$(6.22) \quad \beta = \int_0^1 Z(s, \alpha \chi(s, \alpha), N(1, j(\alpha), j'(\alpha)\beta)) ds, \quad \forall \alpha, \beta.$$

This equation restricts the functions  $Z(\cdot, y, n)$ ,  $\chi$ , and  $j$ . If we set  $\beta = 0$  in (6.22), then we can use (2.10), (2.11), (2.15) to solve the resulting equation for  $j$  in terms of  $\alpha$  and  $\chi(\cdot, \alpha)$ .

We can get alternative representations for  $j$ . Differentiating (6.22) with respect to  $\beta$  we obtain

$$(6.23) \quad j'(\alpha) = \left[ N_Z \int_0^1 Z_n ds \right]^{-1}, \quad \forall \alpha, \beta.$$

Here the arguments of  $N_Z$  and  $Z_n$  are those of  $N$  and  $Z$  in (6.22). We adhere to this same convention below. Equation (6.23) (or (6.22)) may be regarded as an ordinary differential equation for  $j$  parametrized by  $\chi, Z$ . The right side of (6.23) is independent of  $\beta$ , so we could set  $\beta = 0$  there for the purpose of determining  $j$ . Conditions (2.12), (2.15) imply that  $j'(\alpha) > 0$ . Thus  $j'(\alpha)\beta$  ranges over  $(-\infty, \infty)$  with  $\beta$ . It follows that (6.22) is equivalent to

$$(6.24) \quad z = j'(\alpha) \int_0^1 Z(s, \alpha \chi(s, \alpha), N(1, j(\alpha), z)) ds, \quad \forall \alpha, z$$

and to

$$(6.25) \quad Z(1, j(\alpha), n) = j'(\alpha) \int_0^1 Z(s, \alpha \chi(s, \alpha), n) ds, \quad \forall \alpha, n$$

by the remarks leading to (2.13). The substitution of (6.23) into (6.25) yields

$$(6.26) \quad \frac{Z(1, j(\alpha), n)}{Z_n(1, j(\alpha), N(1, j(\alpha), 0))} = \frac{\int_0^1 Z(s, \alpha \chi(s, \alpha), n) ds}{\int_0^1 Z_n(s, \alpha \chi(s, \alpha), N(1, j(\alpha), 0)) ds} \quad \forall \alpha \in \mathbb{R}.$$

This equation would be an identity if  $Z_x = 0$ ,  $j(\alpha) = \alpha$ ,  $\chi(s, \alpha) = 1$ . Thus this equation suggests how  $\chi$  must compensate, if possible, for nonuniformity of material response. We get a collection of necessary conditions for  $Z(\cdot, y, n)$  and  $\chi$  to satisfy any version of (6.22) by differentiating it repeatedly with respect to  $\alpha$  and  $\beta$  (or equivalent variables) and then setting  $\alpha = 0$  and  $\beta$  equal to any convenient value. In this process we can generate a power series for  $j$  and we recover results like (6.15) - (6.19).



## 7. Comments

We could easily treat problems in which the parameter  $\epsilon$  appears elsewhere, provided its vanishing does not contribute to a further reduction of the order of the system. But our methods cannot handle the important problem in which the modulus of viscosity  $N_2$  would go to zero with the parameter  $\epsilon$ . We could, however, treat problems in which the spring is slightly nonuniform, with  $\epsilon$  also serving as a measure of nonuniformity. In this case, the reduced problem would be that for a uniform spring. (We would thereby lose at this level the interesting phenomena of hysteresis generated by nonuniformity.) Of course, to treat small nonuniformity in a truly satisfactory manner, we should properly characterize it by an independent parameter, and then study the asymptotic behavior of solutions on different regions of the neighborhood of the origin in the plane of these two parameters.

Our entire formal theory can be carried out if (2.8) is replaced by a constitutive equation in which  $n(s,t)$  depends upon the history of  $w_s(s,\cdot)$  up to time  $t$ . The study of the reduced problem, which devolves on the analysis of (2.20), can be conducted as in Section 4. In particular, if the spring is uniform, then (2.20) for such materials reduces to

$$(7.1) \quad N(w_s(s,\cdot)) = N(w_s(1,\cdot)),$$

which admits a solution in the form  $w_s(s,t) = w_s(1,t)$ . The treatment of such solutions is identical to that given in Sections 3 and 4. The difficulty with materials with memory is that if the constitutive equation lacks a term like that of (2.8) satisfying (2.12), then we cannot call upon the theory of parabolic equations

to justify the expansions for the full problem in [1]. There is a small but illuminating mathematical theory of one-dimensional problems for materials with memory (cf. Renardy, Hrusa, & Nohel [4] for references) that indicates the sort of difficulties we could expect to encounter. For many such materials there is a threshold such that if initial data are smaller than this threshold, then the solution is well-behaved for all time, whereas if the initial data exceed this threshold, then the solution blows up in finite time (i.e., the body suffers a shock). Thus we could surmise that an extension of the results of Section 4 and of [1] to such materials would require at the least some restrictions on the size of initial data. (Hrusa & Renardy [6] have studied a class of materials with memory that comes closer to capturing the strong dissipative mechanism of (2.8), (2.12) than those just described.)

Many rheological theories describing materials with memory are based on the modelling of behavior on the molecular level with systems of springs and dashpots. It is interesting to examine the discrete model in order to discern in microcosm some of the difficulties we have faced.

For simplicity we examine the linear system with just two degrees of freedom shown in Figure 7.2.  $k$  and  $\ell$  denote spring constants,  $b$  and  $c$  denote damping constants, and  $m$  and  $m$  denote masses. Let  $x$  and  $w$  denote the displacements from the rest positions of the masses  $m$  and  $m$ . Then the equations of motion of this system are

$$(7.3) \quad m\ddot{x} + (b+c)\dot{x} - c\dot{w} + (k+\ell)x - \ell w = 0,$$

$$(7.4) \quad m\ddot{w} + c(\dot{w}-\dot{x}) + \ell(w-x) = 0,$$

the superposed dots denoting time derivatives.

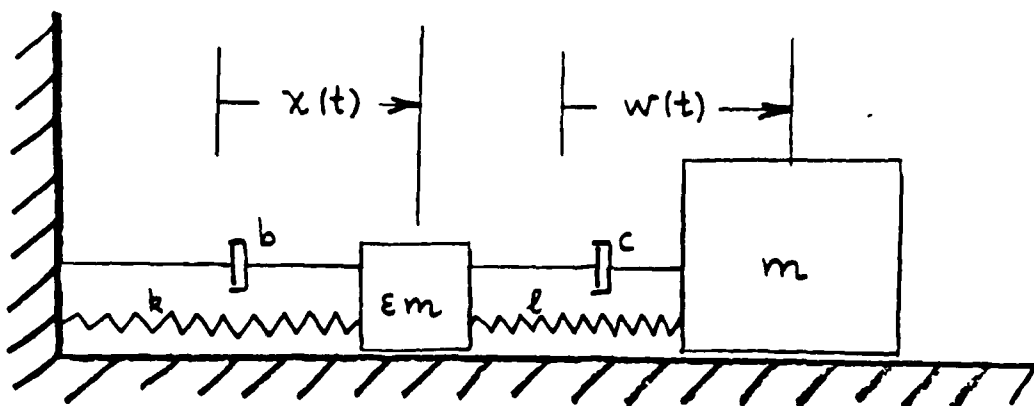


Figure 7.2

For  $\epsilon = 0$ , (7.3) reduces to

$$(7.5) \quad \frac{d}{dt} \left[ x(t) \exp \left( \frac{k+\ell}{b+c} t \right) \right] = \left( \frac{c}{b+c} \right) \exp \left( \frac{k+\ell}{b+c} - \frac{\ell}{c} t \right) \frac{d}{dt} \left[ w(x) \exp \frac{\ell}{c} t \right],$$

which is analogous to (5.3). The solution of this equation for  $x(t)$  in terms of  $w(t)$  is elementary. If  $k/b = \ell/c$ , which corresponds to the constancy of  $r$ , the equation for  $w$  obtained by substituting the representation for  $x$  obtained from (7.5) into (7.4) is a second-order ordinary differential equation. Otherwise, it is a second order functional-differential equation, its memory term reflecting the equivalence of this equation to the third-order system (7.3), (7.4) of ordinary differential equations for  $\epsilon = 0$ .

(In the problems treated in Sections 4 - 6, we reduced a partial differential equation, essentially equivalent to an infinite system of ordinary differential equations, to a single second-order ordinary (functional-) differential equation.) Note the simplifications that follow from (7.3), (7.4) when  $b = 0 = c$ ; these are analogous to the results of Section 3. The full asymptotic analysis of (7.3), (7.4), or of a corresponding system with many degrees of freedom, would follow directly from the asymptotic theory of ordinary differential equations.

These results and those of Sections 4 - 6 raise the following question: Suppose we are given some initial conditions and solutions of an evolution equation in finite-dimensional space. When can we tell whether the equation is an ordinary differential equation of some fixed order? Equivalently, we are given the input and output of a black box. How can we tell whether the dynamics of the black box is governed by ordinary differential equations?

The occurrence of memory effects in elastic wave propagation in nonhomogeneous media is well known (cf. Chen & Gurtin [2], Reiss [7]). The nonhomogeneities reflect signals before they reach the boundaries. These reflections contribute to an effective memory term, the effect being pronounced in composite materials (as Chen & Gurtin show). For our problem, memory effects are not manifested in elastic springs, but only in springs with viscous dissipation.

In 1911, Timoshenko (cf. [9, Section 4.9]) developed a simple technique to account for the mass of a linear spring. Our results show how much richer the theory becomes when viscosity and nonlinearity are allowed to intervene.

Acknowledgment. The research reported here was supported in part by NSF Grant DMS-85-03317 and by AFOSR-URI Grant 87-0073. This project was begun at the Institute for Mathematics and its Applications of the University of Minnesota in 1985.

### References

- OFFICE OF SCIENTIFIC RESEARCH (AFSC)  
1215 N. 17TH AVE  
AFWAL, WISCONSIN 53190-1215  
AFSC Form 323, Rev. 10-79 and is  
available from AFSC Form 323-12.  
1215 N. 17TH AVE  
AFWAL, WISCONSIN 53190-1215  
AFSC Form 323-12  
AFSC Form 323-12  
AFSC Form 323-12
- [1] Antman, S.S., Asymptotic analysis of quasilinear parabolic equations describing viscoelastic bars, to appear (1988).
  - [2] Chen, P.J. and Gurtin, M.E., On the propagation of one-dimensional acceleration waves in laminated composites, J. Appl. Mech. 40 (1973), pp. 1055-1060.
  - [3] Coddington, E.A., and Levinson, N., Theory of Ordinary Differential Equations, 1955, McGraw-Hill.
  - [4] Hale, J.K., Ordinary Differential Equations, 1969, Wiley Interscience.
  - [5] Hille, E., Lectures on Ordinary Differential Equations, 1969, Addison-Wesley.
  - [6] Hrusa, W.J., and Renardy, M., On a class of quasilinear partial integrodifferential equations with singular kernels, J. Diff. Eqs. 64 (1986), pp. 195-220.
  - [7] Reiss, E.L., The impact problem for the Klein-Gordon equation, S.I.A.M. J. Appl. Math. 17 (1969), pp. 526-542.
  - [8] Renardy, M., Hrusa, W.J., and Nohel, J., Mathematical Problems in Viscoelasticity, 1987, Longmans.
  - [9] Timoshenko, S.P., Vibration Problems in Engineering, Third Edition, 1955, Van Nostrand.

END

12-87

DTIC